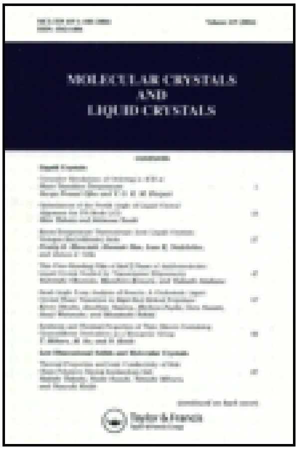

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Analytic 4×4 Propagation Matrices for Linear Optical Media

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A new and efficient 4×4 matrix formalism for light propagation in general linear optical media such as lossy biaxial and optically active materials is developed. The propagation of light is described in terms of the amplitudes of the four propagating modes. Closed form analytic expressions are obtained for the 4×4 propagation matrices, based on exact solutions for the eigenmodes in general linear media.

Keywords: 4×4 matrices; propagation matrices; linear optical media; lossy; biaxial; optical activity

INTRODUCTION

A 4×4 propagation matrix method, introduced by Teitler and Henvis[1], was applied to liquid crystals by Berreman[2] . H.Wohler gave an exact analytical expression, based on the Cayley-Hamilton theorem, for uniaxial media[3], and C.J Chen et.al. recently proposed an iterative numerical scheme to extend this method to biaxial media[4] . In this paper, we propose a 4×4 propagation matrix scheme using a different representation, and provide an analytic expression for the propagation matrix. Our approach uses explicit analytical solutions[5] of Maxwell's equations to propagate eigenmodes in general linear homogeneous media. This enables the closed form construction of our 4×4 propagation matrix, which describes light propagation in lossy biaxial slabs. Our representation is the four-vector of the normal mode amplitudes, rather than the components of the in-plane **E** and **H** vectors. In numerical process, an orthogonality relation is used to make the process efficient.

FORMALISM

Propagating modes in a homogeneous slab

For light obliquely incident on a dielectric slab, typically the component of **k** in the plane of the slab is known rather than the direction of the wave vector **k**. We choose real orthogonal coordinates so that **z** is normal to the slabsurface and **x** is in the plane of incidence. We write $\mathbf{k}/(\omega/c) = (k_x, 0, k_z)$. The in-plane component k_x is same for all propagating modes. The normal component k_z is obtained from the analytical solutions of Maxwell's equations[5][9], which gives for the case of dielectric tensor independent of **k**

$$c_4 k_z^4 + c_3 k_z^3 + c_2 k_z^2 + c\, k_z + c_0 = 0 \tag{1}$$

where

$$c_4 = \mathbf{z}\bar{\bar{\varepsilon}}\mathbf{z}$$

$$c_3 = k_x(\mathbf{x}\bar{\bar{\varepsilon}}\mathbf{z} + \mathbf{z}\bar{\bar{\varepsilon}}\mathbf{x})$$

$$c_2 = k_x^{\ 2}(\mathbf{x}\bar{\bar{\varepsilon}}\mathbf{x} + \mathbf{z}\bar{\bar{\varepsilon}}\mathbf{z}) + \mathbf{z}\bar{\bar{\varepsilon}}^2\mathbf{z} - \mathbf{z}\bar{\bar{\varepsilon}}\mathbf{z}\,\mathrm{tr}(\bar{\bar{\varepsilon}})$$

$$c_1 = k_x^{\ 3}(\mathbf{x}\bar{\bar{\varepsilon}}\mathbf{z} + \mathbf{z}\bar{\bar{\varepsilon}}\mathbf{x}) + k_x(\mathbf{x}\bar{\bar{\varepsilon}}^2\mathbf{z} + \mathbf{z}\bar{\bar{\varepsilon}}^2\mathbf{x} - (\mathbf{x}\bar{\bar{\varepsilon}}\mathbf{z} + \mathbf{z}\bar{\bar{\varepsilon}}\mathbf{x})\mathrm{tr}(\bar{\bar{\varepsilon}}))$$

$$c_0 = k_x^{\ 4}\mathbf{x}\bar{\bar{\varepsilon}}\mathbf{x} + k_x^{\ 2}(\mathbf{x}\bar{\bar{\varepsilon}}^2\mathbf{x} - \mathbf{x}\bar{\bar{\varepsilon}}\mathbf{x}\,\mathrm{tr}(\bar{\bar{\varepsilon}})) + \det(\bar{\bar{\varepsilon}})$$

The propagating eigenmodes satisfy:

$$\mathbf{D} \bullet \mathbf{k} = 0$$

$$\frac{\mathbf{D}\bullet\hat{\mathbf{n}}}{\mathbf{D}\bullet\hat{\mathbf{m}}}=\frac{\hat{\mathbf{n}}\overline{\overline{\varepsilon}}^{-1}\hat{\mathbf{m}}}{\lambda^2-\hat{\mathbf{n}}\overline{\overline{\varepsilon}}^{-1}\hat{\mathbf{n}}},\quad\text{or}\quad\frac{\mathbf{D}\bullet\hat{\mathbf{n}}}{\mathbf{D}\bullet\hat{\mathbf{m}}}=\frac{\lambda^2-\hat{\mathbf{m}}\overline{\overline{\varepsilon}}^{-1}\hat{\mathbf{m}}}{\hat{\mathbf{m}}\overline{\overline{\varepsilon}}^{-1}\hat{\mathbf{n}}}\tag{2}$$

where $\hat{\mathbf{m}}$ and $\hat{\mathbf{n}}$ are arbitary unit vectors with relations to $\hat{\mathbf{k}}$: $\hat{\mathbf{k}}\times\hat{\mathbf{m}}=\hat{\mathbf{n}}$ and $\hat{\mathbf{m}}\times\hat{\mathbf{n}}=\hat{\mathbf{k}}$, $\lambda=\frac{\omega}{ck}$. Usually we select $\hat{\mathbf{m}}=\hat{\mathbf{y}}$, and $\hat{\mathbf{n}}=\hat{\mathbf{k}}\times\hat{\mathbf{y}}$.

$$k_{z1,2}=\frac{-(\frac{a}{2}+e)\pm\sqrt{(\frac{a}{2}+e)^2-4(\frac{y}{2}+f)}}{2}\tag{3}$$

$$k_{z3,4}=\frac{-(\frac{a}{2}-e)\pm\sqrt{(\frac{a}{2}-e)^2-4(\frac{y}{2}-f)}}{2}\tag{4}$$

Solving the quartic Eq.1, one obtains the normal components of the wave vectors of the four modes

where the subscripts denote the normal modes, and the constants are given by

$$a=\frac{c_3}{c_4},\qquad b=\frac{c_2}{c_4},$$

$$c=\frac{c_1}{c_4},\qquad d=\frac{c_0}{c_4},$$

$$p=ac-4d-\frac{b^2}{3},\qquad q=\frac{abc}{3}+\frac{8bd}{3}-c^2-\frac{2b^3}{27}-a^2d$$

$$\phi=\arccos\left(\frac{\sqrt{27}q}{2p\sqrt{-p}}\right),\qquad y=2\sqrt{\frac{-p}{3}}\cos\frac{\phi}{3}+\frac{b}{3},$$

$$e=\sqrt{\frac{a^2}{4}-b+y},\qquad f=\sqrt{\frac{y^2}{4}-d}$$

Construction of the propagation matrices

Substituting $\mathbf{k}_{1,2,3,4}$ into Eq.2 and using

$$\mathbf{E}=\overline{\overline{\varepsilon}}^{-1}\mathbf{D}$$

one obtains the unit eigenvectors for **E**

$$\hat{\mathbf{E}}_{1,2,3,4} = (e_x\mathbf{x} + e_y\mathbf{y} + e_z\mathbf{z})_{1,2,3,4}, \quad \mathbf{E} = E\hat{\mathbf{E}}$$

The magnetic field is given by:

$$\mathbf{H}_{1,2,3,4} = \left(\frac{\mathbf{k}\times\mathbf{E}}{\mu\omega}\right)_{1,2,3,4} = [E(h_x\hat{\mathbf{x}} + h_y\hat{\mathbf{y}} + h_z\hat{\mathbf{z}})]_{1,2,3,4}$$

Our state variable $\tilde{\mathbf{E}} = (E_1, E_2, E_3, E_4)$ is the normal mode amplitude vector. We then construct the propagation matrices:

$$M_j = \begin{bmatrix} e_{x1} & e_{x2} & e_{x3} & e_{x4} \\ e_{y1} & e_{y2} & e_{y3} & e_{y4} \\ h_{x1} & h_{x2} & h_{x3} & h_{x4} \\ h_{y1} & h_{y2} & h_{y3} & h_{y4} \end{bmatrix}, \tag{5}$$

$$\text{and} \quad P_j = \begin{bmatrix} e^{ik_{z1}d_j} & & & 0 \\ & e^{ik_{z2}d_j} & & \\ & & e^{ik_{z3}d_j} & \\ 0 & & & e^{ik_{z4}d_j} \end{bmatrix} \tag{6}$$

where M ensures the continuity of the tangential field components, and P propagates the normal modes across the slab; j denotes the j^{th} slab, with thickness d_j. The propagation relation follows:

$$\tilde{\mathbf{E}}_j = P_j M_j^{-1} M_{j-1} \tilde{\mathbf{E}}_{j-1} \tag{7}$$

For inhomogeneous media with one dimensional variation of dielectric tensor, discretization into n slabs gives:

$$\tilde{E}_{out} = M_{out}^{-1}(\prod_{1}^{n} M_j P_j M_j^{-1}) M_{in} \tilde{E}_{in} \tag{8}$$

Orthogonality Relations and Increasing Computational Efficiency

Oldano had discussed the orthogonality relation between propagating modes[7] for lossless case, we have obtained a different and more general orthogonality relation for fields in a medium with a symmetric dielectric tensor:

$$(\mathbf{E}_i \times \mathbf{H}_j + \mathbf{E}_j \times \mathbf{H}_i) \bullet (\mathbf{k}_i - \mathbf{k}_j) = 0 \tag{9}$$

For a homogeneous slab , the tangential components of the wave vectors of all propagating modes are equal, so Eq.9 can be written as

$$(\mathbf{E}_i \times \mathbf{H}_j + \mathbf{E}_j \times \mathbf{H}_i) \bullet \mathbf{z} = c_{ij} \delta_{ij} \tag{10}$$

Incorporating this relation into the propagation matrix, one obtains

$$M'TM = [c_{ij}\delta_{ij}] \tag{11}$$

where M' is the transpose of M, $T = \begin{bmatrix} 0 & & & 1 \\ & & -1 & \\ & -1 & & \\ 1 & & & 0 \end{bmatrix}$,

$[c_{ij}\delta_{ij}]$ is diagonal with $c_{ii} = 2(e_{xi}h_{yi} - e_{yi}h_{xi})$, for i = 1,2,3,4

then

$$M^{-1} = [c_{ij}\delta_{ij}]^{-1} M'T = \begin{bmatrix} h_{y1}/c_{11} & -h_{x1}/c_{11} & -e_{y1}/c_{11} & e_{x1}/c_{11} \\ h_{y2}/c_{22} & -h_{x2}/c_{22} & -e_{y2}/c_{22} & e_{x2}/c_{22} \\ h_{y3}/c_{33} & -h_{x3}/c_{33} & -e_{y3}/c_{33} & e_{x3}/c_{33} \\ h_{y4}/c_{44} & -h_{x4}/c_{44} & -e_{y4}/c_{44} & e_{x4}/c_{44} \end{bmatrix} \tag{12}$$

If dielectric tensor is not symmetric, Eq. 12 does not hold, and the more general method needs to be used to obtain M^{-1}.

NUMERICAL RESULTS

The transmittance of a Waveguide Based Liquid Crystal (WGLCD) cell with total internal reflection was calculated. Our results, shown in Fig. 1, agree with experiments.[10]

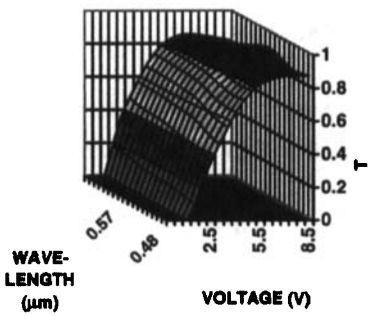

FIGURE 1. Calculated transmittance of a 90° TN WGLCD. thickness d=5µm, refractive index of LC: n_o=1.4673, n_e=1.5185, substrate: n_s=1.5, light dispersion: vertical 15°, horizontal 45°.

We have also calculated the optical rotation of a MO layer with thickness d=113nm, incident light linearly polarized at 20° and dielectric tensor :

$$\overline{\overline{\varepsilon}} = \begin{bmatrix} 7.44+2.65i & -0.029+0.152i & 0 \\ 0.029-0.152i & 7.44+2.65i & 0 \\ 0 & 0 & 7.44+2.65i \end{bmatrix}$$

Our results, shown in Fig. 2, agree exactly with measured values[8].

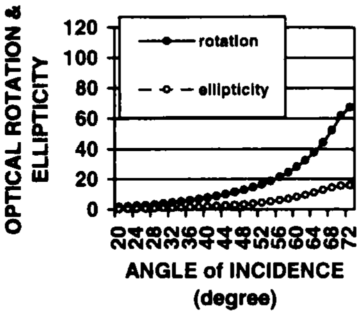

FIGURE 2. Calculated optical rotation and ellipticity as function of angle of incidence of a MO layer

DISCUSSION OF NUMERICAL EFFICIENCY

Since our method uses analytical results without iteration , and makes use of the orthogonality relations in the case of a symmetric dielectric tensor, it is the most efficient of the methods considered.

We used five different numerical schemes to calculate the optical properties of the planar cholesteric cell with the following parameters: thickness d=25p where p is the pitch, the refractive indices $(n_o+n_e)/2=1.5$, $\Delta n=0.07$, $n_g=1.5$ for the substrates . All results agree with the analytical result in the literature[6] .

The running time is shown Fig. 3. 'W' and 'O' represent Wohler's and Oldano's conventional 4X4 schemes [3][7] for uniaxial media . Our formalism was implemented in three different schemes. 'G', doesn't use orthogonality relations, for the most general case, while 'B' and 'U' use orthogonality relations in the case of biaxial and uniaxial materials with a symmetric dielectric tensor. In the 'U' scheme, a quadratic equation is solved rather than the quartic in the 'B' and 'G' schemes.

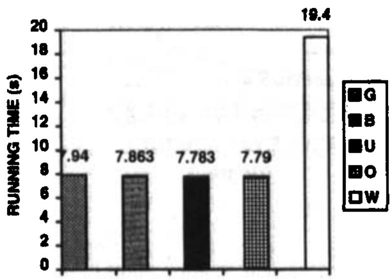

Fig.3 Running times for different computational schemes.

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